

OPTIMAL RESERVE PRICE FOR THE GENERALIZED SECOND-PRICE AUCTION IN SPONSORED SEARCH ADVERTISING

Baichun Xiao
Department of Management, School of Business
Long Island University at C.W. Post,
Brookville, NY 11548, USA
baichun.xiao@liu.edu

Wei Yang
Department of Management, School of Business
Long Island University at C.W. Post,
Brookville, NY 11548, USA
wei.yang@liu.edu

Jun Li
School of Economics and Management
Southwest Jiaotong University, ChengDu 610031 China
lijun@home.swjtu.edu.cn

ABSTRACT

In sponsored search advertising advertisers bid ad links pertaining to a keyword from search engines. This paper presents a pricing model for sponsored search advertising in a dynamic framework. It focuses on the generalized second-price auction, which is widely used by major search engines including Google and Yahoo!. Unlike the assertion in the literature that the number of advertisers and the number of ad links have no impact on the selection of reserve price, our result is noticeably different. We show that the optimal reserve price is affected by both factors. In particular, under a set of mild conditions, the optimal reserve price is equal to the expected value of some order statistic of advertiser's per-click values. Simulations based on the continuous-time bidding process confirm our theoretical findings.

Keywords: sponsored search advertising; locally envy-free equilibrium; revenue management; generalized second-price auction; reserve price.

1. Introduction

Internet advertising has grown rapidly over the last decade. Among various types of internet advertising (e.g., banner, rich media, etc.) sponsored search advertising is one of the fastest growing sources of revenue. Google reported a total revenue of \$16.6 billion over the fiscal year of 2007, and sponsored search advertisement contributed \$16.4 billion [Google 2008].

Early internet advertisements are sold on a per-impression basis. Advertisers pay flat fees for a fixed number of units (usually in thousand impressions). The fees are negotiated on a case-by-case basis. This mechanism is replaced by the generalized first-price (GFP) auctions for sponsored search advertising introduced in 1997. In GFP auctions, each advertiser submits a bid for a particular keyword. When an internet user clicks on the ad link associated with the keyword, the advertiser is charged for what he bids. The method is easy to use and the entry cost is low for advertisers. However, there is no pure strategy equilibrium for the GFP and advertisers reacting to other bidders' moves promptly have a significant advantage. The mechanism encourages insufficient investment and results in volatile bidding prices [Edelman and Ostrovsky 2007]. To remedy the inefficiency, Google introduced a new system in 2002, called AdWords Select. The system stipulates that an advertiser in position i pays a price per click that equals the bidding price of the advertiser in position $i+1$ plus a minimum increment. The mechanism is called the generalized second-price (GSP) auction. It has become a dominant mechanism in sponsored search advertising ever since.

For a certain keyword, Google provides each advertiser a reference price per click to win the first rank before he bids his maximum willingness to pay, or cost per click (CPC). After the bid, the advertiser receives information of

the anticipated number of clicks and an opaque rank of his bid. The rank is opaque because it is estimated based on current market condition including the pool of advertisers and their bid prices. As ad positions in the bottom gain few or no clicks, the capacity of sponsored search advertising is limited. Figure 1 shows the front page of Google when the keyword "Hybrid Vehicle" is entered. There are 3 links on the top and 8 links on the right side. In general, Google's first eleven ranks are most valuable to the advertiser. Unless the advertiser's bid is among the top eleven, his ad will not be displayed in the first page. If a coming advertiser offers a price higher than one of the occupants, the last position on current list will be displaced.

Search engines often use reserve prices to sway advertisers' bidding. When a reserve price is imposed, any bid below the screening level is not admitted to the auction. As reserve price is easy to adjust, it has become a common tool for search engines to manage revenue. Obviously if the reserve price is set too high, it will turn away advertisers and adversely affect the revenue stream because unoccupied ad positions are wasted. On the other hand, if the reserve price is too low, search engines may forgo the opportunity of lifting revenue when advertisers are willing to pay more. Naturally the optimal level of reserve price based on demand, the number of ad positions and other factors is of a great concern to search engines. There have been limited studies on the optimal reserve price in the sponsored search advertising literature. Among other limitations, these studies assume a fixed number of bidders and claim that the optimal reserve price is only dependent on the distribution of bidders' per-click value. Since 2008 Yahoo! has no longer used a fixed minimum bid of \$0.10 and started imposing variable minimum bids for some keywords [Yahoo! Search Marketing Help 2009]. Later Google introduced the first page bid estimate to approximate the minimum CPC needed to show a sponsored link in the first page [Google Adwords Help 2009]. Both minimum bid and first page bid estimate are posted and updated regularly. The primary interest of this research, hence, is to investigate what is the optimal reserve price for the search engine to maximize its expected revenue rate from each sponsored search auction if a fixed reserve price is no longer used in practice.

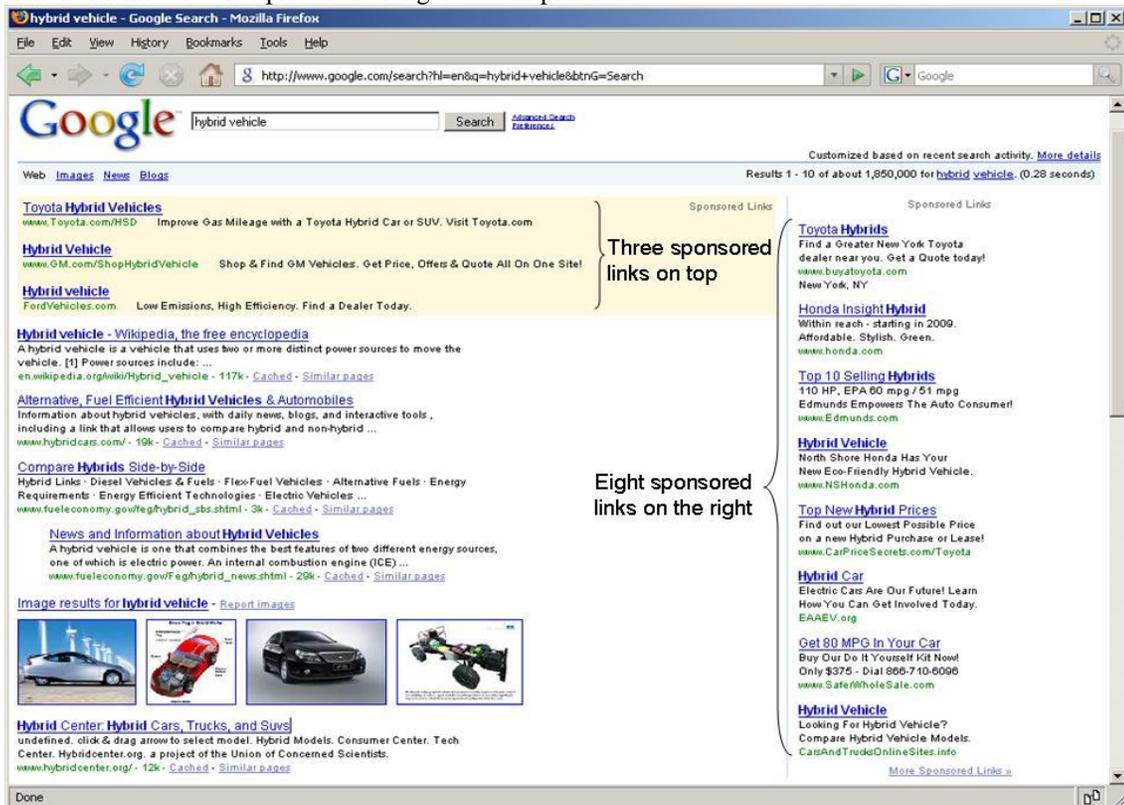


Figure 1: Sponsored advertising links for keyword "Hybrid Vehicle" in Google (Snapshot created on Jan. 6th, 2009)

1.1. Literature Review

Much of the auction literature studies how various factors affect the seller's revenue ranging from winner determination rule, bidders' preference to reserve price. Myerson [1981], Riley and Samuelson [1981] study single-item auctions. They discover that the item should be awarded to the bidder who has the highest willingness to pay if and only if it exceeds a critical reservation value. Maskin and Riley [1989] extend the analysis to auctions with

multiple identical items. Engelbrecht-Wiggans [1987] shows that if the number of bidders is fixed, there exists an optimal reserve price for single-item auctions. When the number of bidders varies, however, decreases in the expected number of bidders may outweigh benefits resulting from the reserve price. Hence, a zero reserve price may become desirable in this case. Bulow and Roberts [1989] adopt the marginal revenue analysis to associate auction theory with economic fundamentals, using a more intuitive method to derive the optimal reserve price presented in Myerson [1981], and Riley and Samuelson [1981]. They let $F(\cdot)$ and $f(\cdot)$ refer to the probability distribution and density functions of bidders' willingness to pay, respectively. They show that if the monopolistic seller sets a take-it-or-leave-it price r , then the expected quantity of sales would be $q(r) = 1 - F(r)$. The seller should disqualify those bidders with negative marginal revenues, and the value that renders zero marginal revenue is the optimal reserve price. Explicitly, it is the solution of

$$MR(r) = \frac{d}{dq} [r \cdot q(r)] = r - \frac{1 - F(r)}{f(r)} = 0. \quad (1)$$

Recent theoretical and empirical works related to sponsored search auctions can be found in Asdemir [2006], Aggarwal and Hartline [2005], Mehta et al. [2005], Szymanski and Lee [2006], and Börgers et al. [2006]. These papers develop near-optimal mechanisms for pricing and capacity allocation. Feng et al. [2007a] compare several auction mechanisms for ad slots through numerical tests. All experiments, however, are conducted for static auctions that are completed in one shot. Edelman et al. [2007] investigate GSP auctions used by search engines to sell sponsored links. They find that a GSP auction generally does not have equilibrium in dominant strategies, and truth-telling is not an equilibrium strategy. They introduce the locally envy-free equilibrium of a simultaneous game induced by GSP where each advertiser cannot be better off by swapping bids with the advertiser ranked one position above him. Similar results can be found in an independent work by Varian [2007]. He points out that the ad position auction is closely related to the assignment game studied in the literature [Roth and Sotomayor 1990], and its equilibrium can be explicitly calculated.

Edelman and Schwarz [2006] study the impact of reserve price and the role of market depth in GSP auctions. They reaffirm the optimality condition (1) and claim that the optimal reserve price is independent of the number of bidders. Feng et al. [2007b] show that in a simultaneous pooled auction if the value function of bidders can be defined as a product of two functions where each relates to the ad position or value per click separately, the optimal reserve price for the search engine is independent of the number of bidders n , or the number of ad positions k . Numerically they show that it is not necessarily optimal to set the k th highest price as the reserve price.

1.2. Motivation and Contribution

To our knowledge, factors around reserve price for sponsored search advertisement have not been fully addressed in the literature. Carried out on a continuous-time basis, sponsored search auctions are infinitely repeated sequential games with very complex equilibrium. To simplify the problem, most theoretical and empirical studies in the literature focus on a snapshot of a "typical" time period. The snapshot assumes each time interval has the same initial condition and bidding parameters. The same number of advertisers simultaneously bid the same total number of ad positions available to them. For example, several studies [Edelman and Schwarz 2006; Feng et al. 2007b] show that the optimal reserve price r in a sponsored search auction should be the same as the one defined in (1). Such an optimal reserve price is only determined by the distribution of value per click, with no connection to the number of bidders or the number of objects on sale. In reality, however, part of or all positions may have been taken when a new bidding is received. Some previous winning bidders are likely to hold their positions unless they run out of budget or are outbid by new comers. Not only competing among themselves, new comers also compete with incumbents with winning positions. The snapshot approach misses the dynamic nature of the auction process and the interdependence between consecutive time periods.

Example 1 Assume that a second-price auction with a single ad position lasts for two periods. There are n bidders in each period whose values per click follow the uniform distribution on $[0, 1]$. If the position is taken in period 1, the winner will stay in the position unless he is outbid in period 2.

According to (1), the best reserve price should be $1/2$. However, it can be verified that when $n = 2$, the optimal reserve price is 0.55 . Furthermore, as n increases, the optimal reserve price also goes up. The reader is referred to the appendix for details. An intuitive explanation is that when there are more advertisers in the auction, the number of bidders who bid over $1/2$ will proportionally increase. As a result, the second highest bid, which is what the winner pays, appears to be greater than $1/2$.

The research conducted in this paper is motivated by the lack of dynamic feature of existing models. We consider advertising capacity as a perishable product and online auction as a continuous-time process. Search

engines try to sell as many available ad positions as possible at the highest prices. We examine how the reserve price affects the seller's expected revenue rate and demonstrate how the optimal reserve price differs from those obtained in the current literature. Our model attempts to maximize the long-term expected revenue rate with reserve price as a control variable. We show that given a total of k ad positions and n advertisers, the optimal reserve price in a GSP auction for sponsored search advertisement is the k th largest expected value per click among advertisers, or its $(n - k + 1)$ th order statistic. In other words, the optimal reserve price depends on the number of ad positions, the number of auction participants in addition to the distribution of their per-click values. Our result differs from what is claimed in Edelman and Schwarz [2006] and others. A simulation is conducted to mimic the dynamic sponsored search auction. Simulation results confirm that the reserve price that equals the k th largest expected per-click value achieves the highest revenue rate for the search engine.

The rest of the paper is organized as follows. In Section 2, we review the GSP auction mechanism for sponsored search advertising. Based on the concept of locally envy-free equilibrium, we derive the optimal reserve price for a GSP auction in Section 3. The section also discusses limitations of the model. Section 4 presents a simulation to mimic the real practice in sponsored search advertising. Concluding remarks are placed in Section 5.

2. Equilibrium in Generalized Second-Price Auction

GSP is the primary mechanism for sponsored search advertising. After a web user submits a search term ("query"), the search engine displays a certain number of relevant sponsored links on result pages in addition to search results. If the user clicks on any of these links, he is directed to an advertiser's web site. The advertiser would be charged by the search engine for sending the user to his web page. Each advertiser has his reservation value on such a click. For the keyword submitted, advertisers specify their maximum willingness to pay for a click by internet users. The position each ad appears on the result page is determined by the rank of bids in a decreasing order. In a GSP auction each advertiser pays the next highest advertiser's bid. There are a limited number of advertising positions that the search engine can provide on each result page, and links placed in different positions have different probabilities to be clicked. In general, a link placed at the top of a page is more likely to be clicked than a link shown at the bottom. However, a user who is directed to the advertiser's web site after clicking the link shown in different positions is assumed to have the same probability to make a purchase. Brooks [2004] shows only moderate differences in purchase probabilities when ads are placed at different positions.

When search engines rank advertisers on the result page, they also need to consider that different advertisers may receive different number of clicks, or click through rates (CTR), even if their ads are placed in the same position. This is because advertisers on top positions may not necessarily generate higher revenue. Yahoo! disregards the difference and simply ranks advertisers in the descending order of their bids. Google ranks bidders according to "rank number", resulted by multiplying each bid with its "quality score." The quality score is based on CTR, relevancy of the keyword and other factors. Each advertiser pays an amount that ensures his rank number to surpass the next advertiser.

Consider a sponsored search auction with k ad positions and n advertisers. Following Edelman et al. [2007], we assume that the distributions of advertisers' per-click values are independent and identical. Advertisers are risk-neutral. The expected CTR received by ad position i is α_i . Since a top ad position receives higher CTR than that of a slot at a lower position, we have $\alpha_i > \alpha_{i+1}$ for $1 \leq i \leq k - 1$ and $\alpha_i = 0$ for $i > k$. Let advertiser j 's bid be b_j and his per-click value be v_j . It is clear that $v_j \geq b_j$. Note that v_j may vary for different advertisers, but it remains the same for bidder j even if his ad is placed in different positions. Let $b^{(i)}$ and $g(i)$ be the bid price and identity of the i th highest bid respectively. If $b^{(i)}$ is greater than the reserve price r , GSP allocates the i th position to the advertiser with the i th highest bid, $g(i)$, where $i \in \{1, \dots, \min(n, k)\}$. When the i th link is clicked by a user, the advertiser $g(i)$ pays the search engine an amount that equals to the next highest bid, $b^{(i+1)}$. Hence, the revenue received by the search engine from the i th position is

$$\pi^{(i)} = \alpha_i b^{(i+1)} \quad (2)$$

and the payoff to advertiser $g(i)$ is

$$p^{(i)} = \alpha_i (v_{g(i)} - b^{(i+1)}) \quad (3)$$

To examine the equilibrium issue in GSP auctions, we begin with the notion defined by Varian [2007].

Definition 1 In a symmetric Nash equilibrium (SNE) bid prices satisfy

$$\alpha_i (v_{g(i)} - b^{(i+1)}) \geq \alpha_j (v_{g(i)} - b^{(j+1)}), \forall i \text{ and } j. \quad (4)$$

Varian [2007] further shows if a set of bid prices satisfies the SNE condition (4) for position $i + 1$ and $i - 1$, then it satisfies (4) for all i . Hence, advertiser $g(i)$ only needs to compare his current position with his two adjacent positions. Since he pays the next highest bid price advertiser $g(i)$ can adjust his bid price $b^{(i)}$ slightly without affecting his position or payment, and there is a range of bid prices that satisfy the SNE condition. Models based on such an equilibrium with various names can be found in Harris and Raviv [1981], Wilson [1993], Varian [2007] and Edelman et al. [2007]. Specifically Edelman et al. [2007] call the SNE with $j = i - 1$ in condition (4) a locally envy-free equilibrium. In this paper we follow the convention and focus on the boundary states of the locally envy-free equilibrium.

Definition 2 In a locally envy-free equilibrium, bid prices satisfy

$$\alpha_{i-1} b^{(i)} = \alpha_i b^{(i+1)} + (\alpha_{i-1} - \alpha_i) v_{g(i)}, \forall i \leq \min(n + 1, k). \quad (5)$$

The locally envy-free equilibrium is intuitive: if the advertiser in position i wants to move up to position $i - 1$, the additional average cost he has to pay should be equivalent to the value of extra clicks he would have received. In such equilibrium all bid prices can be determined explicitly using the recursive formula (5)

In this paper we define a pricing strategy for sponsored search advertising based on the locally envy-free equilibrium with a reserve price. If there are no enough bidders competing for ad positions, i.e., $n \leq k$, then the advertiser in the lowest ranking pays the reserve price; otherwise, the advertiser at the k th position pays the reserve price. Advertisers whose ads appear in higher positions bid according to equation (5). Hence, the level of reserve price set by the search engine directly affects its revenue.

3. The Optimal Reserve Price in Sponsored Search Advertising

3.1. Main Results

To the best of our knowledge, few theoretical analysis of optimal reserve prices consider the dynamic feature and inventory perishability in sponsored search advertising. It is unclear what reserve price maximizes a search engine's expected revenue rate. In this section, we provide a theoretical framework to tackle the problem in a locally envy-free equilibrium.

Let (X_1, X_2, \dots, X_n) be an n -dimensional random vector, and (x_1, x_2, \dots, x_n) be an n -tuple assumed by (X_1, X_2, \dots, X_n) . Rearranging x_1, x_2, \dots, x_n in increasing order so that

$$x_{(1)} \leq x_{(2)} \leq \dots \leq x_{(n)},$$

where $x_{(1)} = \min\{x_1, x_2, \dots, x_n\}$, $x_{(2)}$ is the second smallest value in the n -tuple, and so on. It is clear that $x_{(n)} = \max\{x_1, x_2, \dots, x_n\}$. The function $X_{(k)}$ of (X_1, X_2, \dots, X_n) that takes on the $x_{(k)}$ in each possible sequence (x_1, x_2, \dots, x_n) of values assumed by (X_1, X_2, \dots, X_n) is known as the k th order statistic. Let X_1, X_2, \dots, X_n be identical and independent random variables with common probability density function $f(x)$ and common distribution function $F(x)$. It is well known that the marginal probability density function of $X_{(k)}$ is given by

$$f_{(k)}(x) = \frac{n!}{(k-1)!(n-k)!} [F(x)]^{k-1} [1-F(x)]^{n-k} f(x) \quad (6)$$

and its expectation can be written as

$$E[X_{(k)}] = \int_{-\infty}^{\infty} x f_{(k)}(x) dx = \frac{n!}{(k-1)!(n-k)!} \int_{-\infty}^{\infty} x [F(x)]^{k-1} [1-F(x)]^{n-k} f(x) dx.$$

It is easy to determine the bid price winning the i th ad position $b^{(i)}$ recursively from the locally envy-free equilibrium. Let $n \geq k$. The advertiser at the last position pays the reserve price. For convenience, let $b^{(k+1)} = r$. For $i \leq k$,

$$b^{(i)} = \frac{\alpha_i b^{(i+1)} + (a_{i-1} - \alpha_i) X_{(n-i+1)}}{\alpha_{i-1}}. \tag{7}$$

When there are k positions offered at a reserve price r , revenue to the search engine per unit of time is a random variable given by

$$R(n, k) = r \alpha_k + \sum_{l=1}^{k-1} \alpha_l b^{(l+1)}. \tag{8}$$

In view of (7), with $l+1 \leq k$ and n bidders in the auction we have

$$\begin{aligned} \alpha_l b^{(l+1)} &= \alpha_{l+1} b^{(l+2)} + (\alpha_l - \alpha_{l+1}) X_{(n-l)} \\ &= r \alpha_k + (\alpha_l - \alpha_{l+1}) X_{(n-l)} + \dots + (\alpha_{k-1} - \alpha_k) X_{(n-k+1)}, \end{aligned}$$

which can be substituted into (8) to form a linear function of order statistics:

$$R(n, k) = r k \alpha_k + \sum_{l=1}^{k-1} l (\alpha_l - \alpha_{l+1}) X_{(n-l)}. \tag{9}$$

Since $X_{(n-l)}$ is uncertain, the expected revenue rate can be depicted as

$$\Psi(r) = E[R(n, k)] = r k \alpha_k + \sum_{l=1}^{k-1} l (\alpha_l - \alpha_{l+1}) E[X_{(n-l)}]$$

Lemma 1 Consider a GSP auction for sponsored search advertising with n advertisers and k ad links. Assume that each advertiser's per-click value is independently and identically distributed (i.i.d.). Let r be a reserve price satisfying $r < E[X_{(n-k+1)}]$, then the expected revenue for the search engine is an increasing function of r .

The proof of Lemma 1 and all other proofs are placed in the appendix.

Lemma 1 states that the optimal reserve price should not be set below the expected value of the $(n-k+1)$ th order statistic of advertiser's per-click values. This is intuitively straightforward. If two distinct reserve prices result in the same expected number of ad slots to be sold, then the higher price should lead to more revenues as the k th highest bidder pays more.

Lemma 2 Consider a GSP auction for sponsored search advertising with n advertisers and k ad links. Assume that (i) each advertiser's per-click value is finite and i.i.d.; (ii) $E[X_i] \geq \frac{a}{2}$, where $a > 0$ is the highest per-click value; (iii) $n \geq (k-1) \left(1 + \frac{\alpha_{k-1}}{\alpha_k} \right)$. Then for any r satisfying $E[X_{(n-k+1)}] < r < E[X_{(n-k+2)}]$, r cannot be the optimal reserve price for the search engine.

We note that Lemma 2 needs several conditions. Condition (i) is reasonable as no per-click values can go to infinity. Condition (iii) is very mild, especially when the number of bidders is large compared to the number of

positions available. Condition (ii) clearly holds for any symmetric distribution such as normal and uniform distributions. It also holds for some asymmetric distributions. For example, the β distribution with $\alpha \geq \beta$.

The dilemma faced by the search engine is whether to sell at most $k-1$ links at a higher starting price such as $r > E[X_{(n-k+1)}]$, or sell k links at a lower price. Lemma 2 shows that although raising the reserve price may improve the revenue collected from each position, it fails to do so if the expected number of ad slots sold is compromised. This coincides with the observation by Engelbrecht-Wiggans [1987] who finds that “any decrease in the (expected) number of bidders hurts the bid-taker’s expected revenue more than any benefits from a nontrivial reservation price¹”.

Combining the above two lemmas, we claim the following theorem.

Theorem 1 Assume that each advertiser’s per-click value is i.i.d., then the optimal reserve price for the search engine is $r = E[X_{(n-k+1)}]$.

When the search engine raises the reserve price, it does not only affect the advertiser at the last position, but also have a ripple effect to the rest occupants. This can be learned from equation (7). As r increases, the bidder winning the k th position has to pay higher reserve price. This in turn will affect $b^{(k)}$ in view of the equilibrium condition, and each other bid price $b^{(i)}$. As reserve price rises from zero, the advertiser at the last position has to increase his bid price to avoid elimination. This new price may outbid those above him and render him a higher position. To prevent losing his current position, the advertiser at the second lowest position has to bid higher accordingly, so does the one at the third lowest position, and so on. When the reserve price exceeds some advertiser’s reservation value, he has no choice but to quit the auction, causing revenue loss to the search engine. Hence, the search engine should raise the reserve price high enough while ensuring the sales of all k ad positions. The $(n-k+1)$ th order statistic of per-click value is the optimal reserve price in a GSP auction for sponsored search advertisement. As a special case, the optimal reserve price is $\frac{n-k+1}{n+1}$ if advertisers’ per-click values follow a uniform distribution on $[0,1]$.

3.2. Limitations of the Model

In the previous section we derived the optimal reserve price under certain conditions. The theoretical developments, however, have their limitations. First, although we consider the dynamic nature of various number of advertisers participating the auction and incorporate it into the optimal reserve price, our model fails to capture some other dynamic aspects in the GSP auctions. The GSP auctions in reality are infinite repeated games where the sets of equilibria can be very large and the strategies leading to such equilibria are complex. As Edelman et al. [2007] argue, it is not reasonable to expect advertisers to be able to execute such strategies. Since advertisers can adjust and finalize their bids after reviewing the estimated rank, CTR and other performance feedbacks released by the search engine, following Edelman et al. [2007] and Varian [2007] we focus on a static GSP auction in which bids are stabilized. Secondly, the number of advertisers participating in the auction, although being reflected in the reserve price, is unknown and needs to be estimated. This could be difficult when the number of advertisers changes constantly. Thirdly, our main result covers a wide range of probability distributions including all symmetric distributions, it remains to be seen if the optimal reserve price applies to arbitrary distributions.

We note that advertisers’ budget constraints will also affect the result of our model. For example, when the reserve price is beyond advertisers’ budget, it will affect the number of participants in the auction, which in turn, impact the optimal reserve price. However, this makes the model much more complex and is beyond the scope of this research. We leave it as a future research question.

4. Numerical Experiment

We design a simulation to mimic the real practice in sponsored search advertising. Different from the experiments conducted in Feng et al. [2007a], and Edelman and Schwarz [2006] where only static auction is tested, we simulate a dynamic and continuous auction aiming to answer the following questions: (1) What is the best reserve price in a GSP auction for sponsored search advertisement? (2) What is the revenue benefit to the search engine of setting the reserve price as the $(n-k+1)$ th order statistics of the per-click value?

¹The nontrivial reservation price means the optimal reservation price based on the fixed number of bidders.

4.1. Design of the Simulation

The simulation is conducted over a horizon of $[0, T]$. There are k ad positions available for sale. Instead of assuming an exogenously known number of advertisers, we examine a stream of bid arrivals following a Poisson process with a density λ .

We extend the GSP auction described in Edelman et al. [2007] to a dynamic setting, where the sponsored search auction progresses over the course of time. Each new bid triggers a new round of re-allocation of ad positions. Figure 2 shows the timeline of the simulation. At time t_1 the first bid b_1 is submitted. If it is greater than the reserve price, the first ad position will be awarded. Arriving at time t_j the j th bid competes with current occupants. If there are positions unfilled, one position will be awarded to the new qualified bidder based on his rank. Otherwise the entrant has to outbid one of the current occupants to secure a position. Let $b_{t_j}^{(i)}$ and $g_{t_j}(i)$ be the bid price and identity of the i th highest bid at time t_j . In a locally envy-free equilibrium bid prices should satisfy

$$\alpha_{i-1} b_{t_j}^{(i)} = \alpha_i b_{t_j}^{(i+1)} + (\alpha_{i-1} - \alpha_i) v_{g_{t_j}(i)}, \tag{10}$$

where $i = 1, \dots, \min(j, k)$. Since the CTR for the i th ad position is α_i and the advertiser $g_{t_j}(i)$ pays $b_{t_j}^{(i+1)}$ to the search engine for each click, over the inter-arrival time $\Delta t_j = t_{j+1} - t_j$ the average revenue received by the search engine from the i th position is

$$\pi_{t_j}^{(i)} = \alpha_i b_{t_j}^{(i+1)} \Delta t_j, \tag{11}$$

and the average payoff to advertiser $g_{t_j}(i)$ is

$$p_{t_j}^{(i)} = \alpha_i (v_{g_{t_j}(i)} - b_{t_j}^{(i+1)}) \Delta t_j, \text{ where } b_{t_j}^{(i)} > r. \tag{12}$$

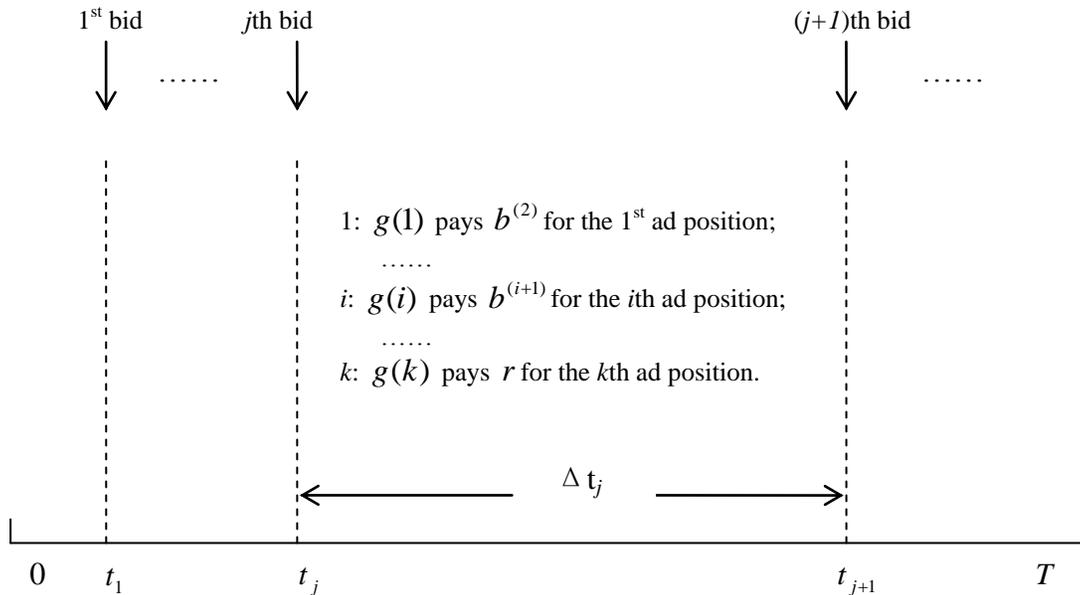


Figure 2: Timeline for the simulation

Let J be the last bid arrival before T where $J = \sup\{J \mid \sum_{j=0}^J \Delta t_j \leq T\}$, where $t_0 = 0$. The weighted average revenue rate received by the search engine over $[0, T]$ is

$$\pi = \frac{1}{T} \sum_{i=1}^k \sum_{j=1}^J \pi_{t_j}^{(i)}, \tag{13}$$

and the weighted average payoff rate to advertisers is

$$p = \frac{1}{T} \sum_{i=1}^k \sum_{j=1}^J p_{t_j}^{(i)}. \tag{14}$$

In the following experiments we use π and p to gauge the impact of reserve price.

4.2. Experiment 1: Comparison of Dynamic and Static GSP Auctions

This experiment compares the performance of a dynamic GSP auction and a static GSP auction when different levels of reserve price are tested. Let $T = 50$ and $k = 8$. To be consistent with the numerical tests in Feng et al. [2007a] we assume a constant decaying attention factor $c = 1.418$ among ad positions, meaning that the i th ad position receives 41.8% more clicks on average than the $(i + 1)$ th position. Following the simulation methodology of Edelman and Schwarz [2006], we assume that the per-click value of each advertiser follows a lognormal distribution with mean $\mu = 1.0$ and standard deviation $\sigma = 0.25$. Reserve prices ranging from 0 to 5 are tested.

Bid arrivals follow a Poisson distribution with density $\lambda = 4$ in a dynamic auction. To simulate a static auction we need to decide a fixed number of advertisers. The value of the number of advertisers in each static auction is estimated from its corresponding instance in the dynamic auction: we divide the testing horizon into T discrete periods. The number of advertisers of each period then equals the number of ad positions allocated to advertisers at the beginning of each period plus the new bid arrivals over that period. The average number of advertisers over T periods gives the number of advertisers in each static auction. All reported results are averaged over 1000 simulated auctions.

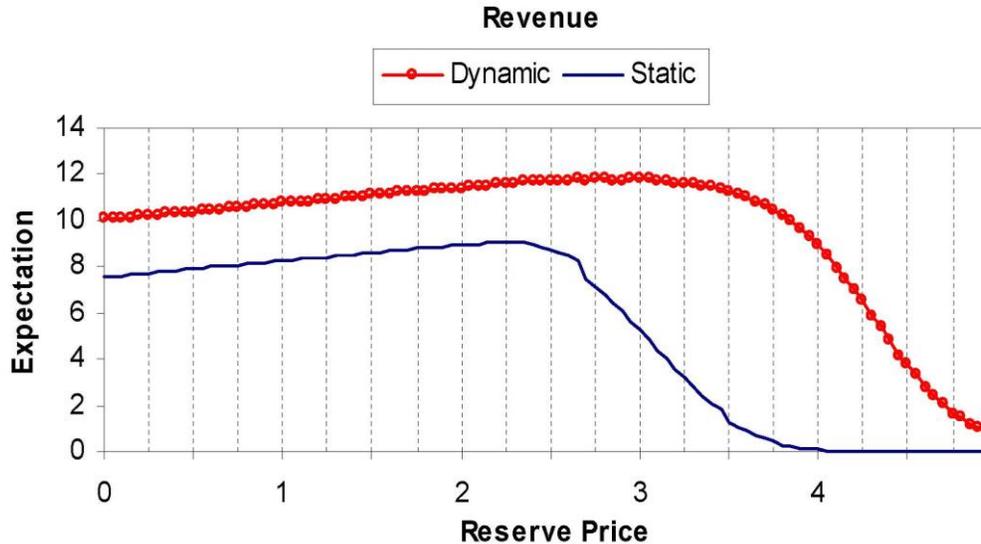


Figure 3: Expected Revenue in Dynamic Auction and Static Auction for Sponsored Search

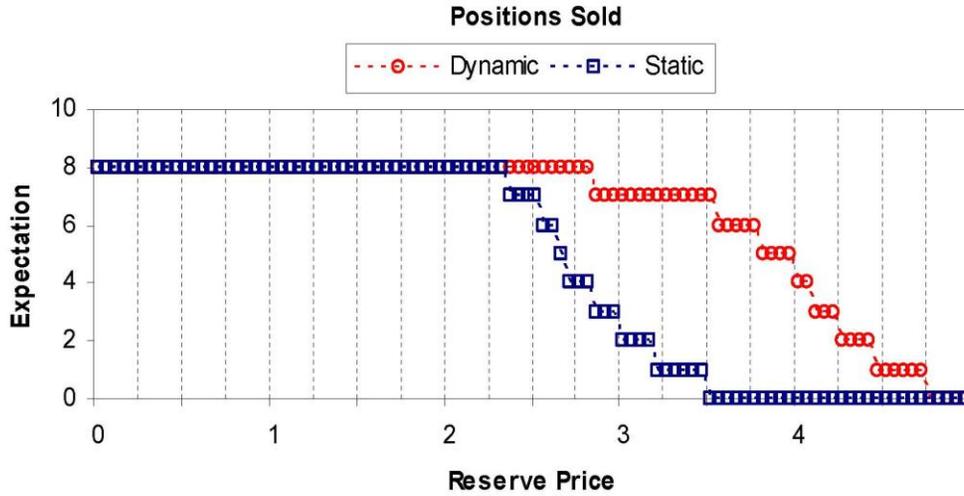


Figure 4: Expected Number of Ad Positions Sold in Dynamic Auction and Static Auction for Sponsored

The expected revenue curves in both dynamic and static auctions are plotted in Figure 3. It suggests that there is an optimal reserve price that maximizes the expected revenue in a dynamic auction for sponsored search advertising, and the level of such a reserve price is higher than the one in a static auction. Figure 4 presents the sales of ad positions. A low reserve price promotes sales; however, it does not create adequate revenue to the seller due to low payment received from advertisers. As the reserve price approaches zero, all eight ad slots are occupied, delivering the highest payoff to advertisers while the lowest revenue to the search engine. On the other hand, if the reserve price is beyond some advertisers' willingness to pay, a few ad positions may be left vacant, resulting in lost sales. In Figure 4, for instance, if the reserve price is set as 4.0 only half ad slots are expected to be allocated in a dynamic auction. Hence, the level of reserve price directly affects the expected revenue.

In the equilibrium of a GSP auction, advertisers with higher per-click values are assigned to top positions while those with lower willingness to pay appear in bottom slots. The equilibrium bid price can be determined via a recursive equation defined in (5). Since there are only k positions for sale, it is optimal for the first excluded advertiser (i.e., the one ranks at $(k + 1)$ th position) to bid his true value. The argument is similar to the Vickrey

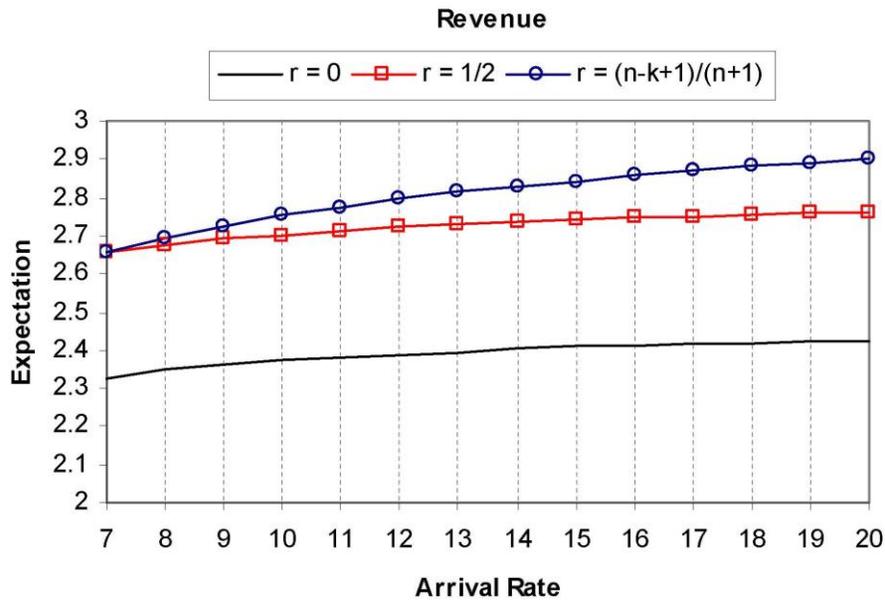


Figure 5: Expected Revenue with Three Reserve Prices Imposed in Dynamic GSP

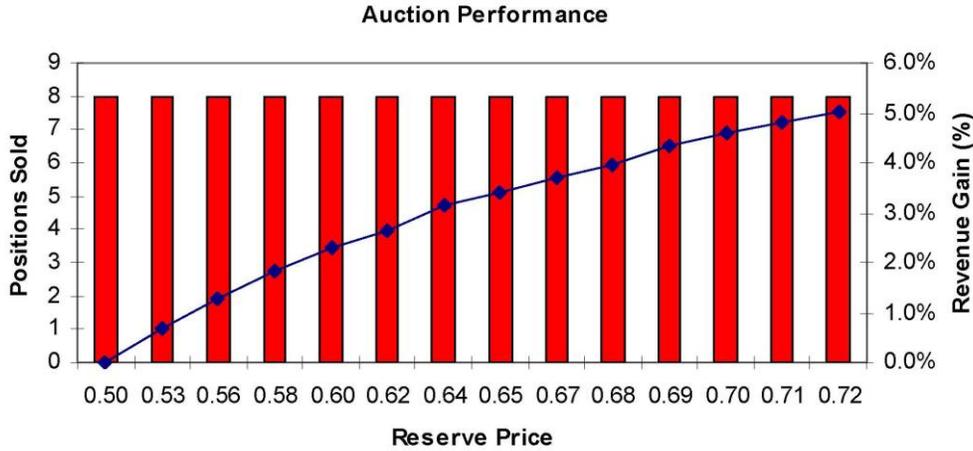


Figure 6: Auction Performance When $r = E[X_{(n-k+1)}]$

auction [Varian 2007]. If the advertiser is excluded, then it is meaningless to bid lower than his value; but if he happens to win a position due to possible exits of higher bidders, then the advertiser is able to make a profit. Since the advertiser occupying the k th slot is supposed to pay the $(k + 1)$ th highest bid price, the search engine should set the reserve price as the k th largest value per click, or the $(n - k + 1)$ th order statistic, to extract the maximal revenue.

4.3. Experiment 2: Optimal Reserve Price in Dynamic GSP Auction

We test three reserve prices: (1) $r = 0$; (2) $r = \frac{1 - F(r)}{f(r)}$ and (3) $r = E[X_{(n-k+1)}]$ in dynamic GSP auctions when the Poisson arrival rate of bids varies from 7 to 20. Other simulation parameters remain the same as in section 4.2 except that the value per click follows a uniform distribution on $[0, 1]$. Thus, three reserve prices under test become (1) $r = 0$; (2) $r = \frac{1}{2}$ and (3) $r = \frac{n - k + 1}{n + 1}$, respectively. This experiment aims to verify which reserve price is the best for sponsored search auctions.

To obtain the expected value of $X_{(n-k+1)}$ we estimate the average number of advertisers of a dynamic auction using the same approach in section 4.2: The simulation horizon is divided into T periods. The value of n is the average number of advertiser of each period, i.e., the sum of total number of ad slots occupied at the beginning of each period and the number of new entrants over that period. Figure 5 plots the expected revenue rate in dynamic GSP auctions when r equals 0, $1/2$ and $\frac{n - k + 1}{n + 1}$, respectively. It is clear that a zero reserve price leads to the lowest revenue rate because each advertiser pays the minimum for one ad slot. When arrival rate $\lambda = 7$, the average number of advertisers is estimated to be $n = 15$ by simulation. Hence, $r = \frac{1}{2}$ and $r = \frac{n - k + 1}{n + 1}$ create the same revenue because they equal to each other when $n = 15$ and $k = 8$. As demand increases, however, using $r = \frac{n - k + 1}{n + 1}$ clearly generates higher revenue than a fixed reserve price that solely depends on the value distribution function.

To further demonstrate why $r = E[X_{(n-k+1)}]$ is the optimal reserve price for sponsored search auction, we depict the sales of ad positions under $r = E[X_{(n-k+1)}]$ and its revenue gain over $r = \frac{1 - F(r)}{f(r)}$ in Figure 6. The horizontal axis represents the corresponding $(n - k + 1)$ th order statistics of value per click as λ varies from 7 to 20. It shows that if the demand for ad slots is high enough, imposing a higher reserve price of $r = E[X_{(n-k+1)}] > \frac{1}{2}$ is able to allocate all perishable inventory and create more revenue for the search engine. As stated in the literature [Feng et al. 2007a; Edelman and Schwarz 2006], when the number of advertisers increases, the impact of reserve price diminishes due to more competition created among bidders. Figure 6 shows that as demand rises, the revenue gain of setting $r = E[X_{(n-k+1)}]$ becomes more significant. For instance, when $\lambda = 20$, using $r = E[X_{(n-k+1)}] = 0.72$ results in 5% more revenue than $r = \frac{1 - F(r)}{f(r)}$. The marginal revenue benefit of setting $r = E[X_{(n-k+1)}]$, however, is decreasing in demand, which is understandable because as n becomes much larger relatively to a fixed k , its impact to the $(n - k + 1)$ th order statistics is negligible.

The results above are consistent with the empirical evidences shown in the US market where the CPC for keywords can range from a few cents to \$100 depending on the popularity [Morrison 2009]. In addition, adopting similar strategy as Google and no longer fixing the minimum bid at \$.10, Yahoo! now uses variable reserve prices in some keyword markets based on several factors such as the number of bidders and the bid amounts [Yahoo! Search Marketing Help 2009]. Such a shift in pricing strategy corroborates our theoretical findings.

5. Conclusion

Sponsored search advertising is a multi-billion dollar business. Search engines such as Google and Yahoo! often use reserve price to influence advertiser's bidding for ad links pertaining to a keyword through the generalized second-price (GSP) auction. However, as advertising capacity is perishable and the auction process is dynamic, arbitrary thresholds may adversely affect revenues.

This paper studies the optimal reserve price in a static GSP auction for sponsored search advertising. We prove that given a total of k ad positions and n advertisers the reserve price that maximizes the search engine's average revenue rate is the k th largest expected value per click, or its $(n-k+1)$ th order statistic among advertisers. Hence, the optimal reserve price in a sponsored search auction depends on the number of ad positions, the number of advertisers and their per-click values. Our result noticeably differs from the existing ones in the literature where the optimal reserve price in sponsored search auction is claimed to be only related to bidders' value distribution.

A simulation mimicking the real practice in sponsored search advertising shows that a constant reserve price fails to capture the dynamics in the auction process. If the demand is large, setting reserve price as $r = E[X_{(n-k+1)}]$ is able to allocate all ad positions and create higher revenue rate than other alternatives. Practical implementation of variable reserve price in Google and Yahoo! corroborates our findings.

Several possible extensions to this work are worth being mentioned. First, the current solution is applicable to symmetric distributions of per-click values and other distributions with certain conditions. One may extend it to a general value distribution, explore the structural property of the optimal reserve price with respect to the number of ad positions and number of bidders. Secondly, the optimal reserve price is derived in a single-shot static GSP auction. It may be interesting to consider a dynamic model investigating how the search engine should update the reserve price based on the incumbent and potential new entry in a multi-period horizon. Finally, the impact of advertisers' budget constraints on the optimal reserve price deserves being examined.

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APPENDIX

Analysis of Example 1:

Let μ_1 and ν_1 be the highest bid in period 1 and 2 respectively. Let the reserve price be r and search engine's revenue in period i be $R_i, i = 1, 2$. In view of Engelbrecht-Wiggans (1987), we have

$$E[R_1] = nr^n(1-r) + n(n-1)(1-r^n) + \frac{n(n-1)}{n+1}(1-r^{n+1})$$

Notice that the highest bidder in period 1 will stay in period 2 and is joined by other $n - 1$ advertisers. If $R_1 = 0$, then $E[R_1] = E[R_2]$. Otherwise we consider the following cases. (i) If $\nu_1 < r$, then $R_2 = r$; (ii) If $r < \nu_1 < \mu_1$, then $R_2 = \nu_1$; (iii) If $\nu_1 > \mu_1$, then $R_2 = \mu_1$. As the pdf of μ_1 and ν_1 are $n\mu_1^{n-1}$ and $(n-1)\nu_1^{n-2}$ respectively, it can be shown that

$$\begin{aligned} E[R_2] &= r^n E[R_1] + (1-r^n) \left\{ r^n + \int_r^{\mu_1} \nu_1 (n-1)\nu_1^{n-2} d\nu_1 + \int_{\mu_1}^1 \mu_1 (n-1)\nu_1^{n-2} d\nu_1 \right\} n\mu_1^{n-1} d\mu_1 \\ &= nr^{2n}(1-r) + n(n-1)r^n(1-r^n) + \frac{n(n-1)}{n+1}r^n(1-r^{n+1}) + \\ &(1-r^n) \left[r^n + \frac{n-1}{2n}(1-r^{2n}) - \frac{n-1}{n}r^n(1-r^n) + \frac{n}{n+1}(1-r^{n+1}) - \frac{1}{2}(1-r^{2n}) \right], \end{aligned}$$

and

$$\begin{aligned} E[R] &= E[R_1 + R_2] \\ &= (1+r^n) \left[nr^n(1-r) + n(n-1)(1-r^n) + \frac{n(n-1)}{n+1}(1-r^{n+1}) \right] + \\ &(1-r^n) \left[r^n + \frac{n-1}{2n}(1-r^{2n}) - \frac{n-1}{n}r^n(1-r^n) + \frac{n}{n+1}(1-r^{n+1}) - \frac{1}{2}(1-r^{2n}) \right]. \end{aligned}$$

As a result,

$$\begin{aligned} \frac{dE[R]}{dr} &= \frac{4n^3 - 2n^2 + 3n + 3}{2(n+1)} r^{n-1} - n(2n+1)r^n + [-2n^3 + 4n^2 + 2n - 3]r^{2n-1} + \\ &\frac{n(4n^2 - 1)}{n+1} r^{2n} + \frac{3(1-2n)}{2} r^{3n-1}. \end{aligned} \tag{15}$$

It can be verified that $\left. \frac{dE[R]}{dr} \right|_{r=1/2} > 0$ and $\left. \frac{dE[R]}{dr} \right|_{r=1} < 0$ for sufficiently large n . Hence, the optimal reserve price is between 1/2 and 1.

Proof of Lemma 1

Let $r_1 < r_2 < E[X_{(n-k+1)}]$ It clear that at both r_1 and r_2 , the expected number of ad positions the search engine can sell is k and the respected revenue rates, represented by $\Psi(r_1)$ and $\Psi(r_2)$, are

$$\begin{aligned} \Psi(r_1) &= r_1 k \alpha_k + \sum_{l=1}^{k-1} l(\alpha_l - \alpha_{l+1}) E[X_{(n-l)}] \text{ and} \\ \Psi(r_2) &= r_2 k \alpha_k + \sum_{l=1}^{k-1} l(\alpha_l - \alpha_{l+1}) E[X_{(n-l)}]. \end{aligned}$$

Hence,

$$\Psi(r_2) - \Psi(r_1) = k\alpha_k(r_2 - r_1) > 0.$$

Proof of Lemma 2

Since $E[X_{(n-k+1)}] < r < E[X_{(n-k+2)}]$, if r is set up as a reserve price, then the expected number of ad positions the search engine can sell is $k - 1$. According to Lemma 1, any r_1 satisfying $r < r_1 < E[X_{(n-k+2)}]$ will generate more expected revenue than r does. Let $r_1 = E[X_{(n-k+2)}]$ and $r_0 = E[X_{(n-k+1)}]$, we show that $\Psi(r_0) \geq \Psi(r_1)$. Note that when the reserve price is r_0 , the expected number of ad slots the search engine can sell is k . Hence, we have

$$\begin{aligned} \Psi(r_0) - \Psi(r_1) &= k\alpha_k E[X_{(n-k+1)}] + \sum_{l=1}^{k-1} l(\alpha_l - \alpha_{l+1}) E[X_{(n-l)}] \\ &\quad - (k-1)\alpha_{k-1} E[X_{(n-k+2)}] - \sum_{l=1}^{k-2} l(\alpha_l - \alpha_{l+1}) E[X_{(n-l)}] \\ &= \alpha_k E[X_{(n-k+1)}] - (k-1)\alpha_{k-1} \{E[X_{(n-k+2)}] - E[X_{(n-k+1)}]\} \\ &= \alpha_k E[X_{(n-k+1)}] \left\{ 1 - \frac{(k-1)\alpha_{k-1}}{\alpha_k} \cdot \frac{E[X_{(n-k+2)}] - E[X_{(n-k+1)}]}{E[X_{(n-k+1)}]} \right\}. \end{aligned}$$

So it suffices to show

$$1 - \frac{(k-1)\alpha_{k-1}}{\alpha_k} \cdot \frac{E[X_{(n-k+2)}] - E[X_{(n-k+1)}]}{E[X_{(n-k+1)}]} \geq 0,$$

or equivalently

$$\frac{E[X_{(n-k+2)}] - E[X_{(n-k+1)}]}{E[X_{(n-k+1)}]} \leq \frac{\alpha_k}{(k-1)\alpha_{k-1}}.$$

It is not difficult to show the following equation after applying integration by parts:

$$E[X_{(n-k+2)}] - E[X_{(n-k+1)}] = \frac{n!}{(n-k+1)!(k-1)!} \int_0^\infty [F(x)]^{n-k+1} [1-F(x)]^{k-1} dx.$$

Note that

$$E[X_{(n-k+1)}] = \frac{n!}{(n-k)!(k-1)!} \int_0^\infty x [F(x)]^{n-k+1} [1-F(x)]^{k-1} f(x) dx.$$

As a result,

$$\frac{E[X_{(n-k+2)}] - E[X_{(n-k+1)}]}{E[X_{(n-k+1)}]} = \frac{1}{n-k+1} \frac{\int_0^\infty [F(x)]^{n-k+1} [1-F(x)]^{k-1} dx}{\int_0^\infty x [F(x)]^{n-k} [1-F(x)]^{k-1} f(x) dx}. \quad (16)$$

We show the right hand of (16) is less than $\frac{1}{n-k+1}$, or

$$\frac{\int_0^a [F(x)]^{n-k+1} [1-F(x)]^{k-1} dx}{\int_0^a x[F(x)]^{n-k} [1-F(x)]^{k-1} f(x) dx} \leq 1,$$

where a is the largest per-click value among advertisers. Observe

$$\begin{aligned} & \int_0^a x[F(x)]^{n-k} [1-F(x)]^{k-1} f(x) dx - \int_0^a [F(x)]^{n-k+1} [1-F(x)]^{k-1} dx \\ &= \int_0^a \left\{ x[F(x)]^{n-k} [1-F(x)]^{k-1} f(x) - [F(x)]^{n-k+1} [1-F(x)]^{k-1} \right\} dx \\ &= [F(\xi)]^{n-k} [1-F(\xi)]^{k-1} \int_0^a [xf(x) - F(x)] dx \\ &= [F(\xi)]^{n-k} [1-F(\xi)]^{k-1} \left[E(X) - \int_0^a F(x) dx \right] \\ &= [F(\xi)]^{n-k} [1-F(\xi)]^{k-1} [E(X) - a + E(X)] \\ &\geq 0, \end{aligned}$$

where $\xi \in (0, a)$ and the last inequality results from the assumption that $E(X) \geq \frac{a}{2}$. Thus, if

$$\frac{1}{n-k+1} < \frac{\alpha_k}{(k-1)\alpha_{k-1}},$$

or

$$n \geq (k-1) \left(1 + \frac{\alpha_{k-1}}{\alpha_k} \right),$$

we have $\Psi(r_0) \geq \Psi(r_1) > \Psi(r)$.